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Coloring of Double Disk Graphs

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Abstract. We study the problem of minimizing the number of colors for vertex-coloring of double disk graphs and in this note, show a polynomial-time 31-approximation for the problem, which improves an existing result.

1. Introduction

Consider n points $v_1, \ldots, v_n$ on the Euclidean plane, each with two disks centered at the point. Let r_i and R_i with $r_i \leqslant R_i$ be radiuses of the two disks at point v_i. Define a graph G with vertex set $\{v_1, \ldots, v_n\}$ and edge (v_i, v_j) exists if and only if $d(v_i, v_j) \leqslant \max(R_i + r_j, r_i + R_j)$, that is, either outer disk of v_i intersects with inner disk of v_j or inner disk of v_i intersects with outer disk of v_j. Such a graph G is called a *double disk graph*. The double disk graph has application in wireless communication [3]. Motivated from frequency assignment problem in wireless communication, Malesinska et al. [3] studied the problem of minimizing the number of colors for vertex-coloring of double disk graphs. The problem is NP-hard since the problem on unit disk graphs is NP-hard [3] and the unit disk graph is a special case of the double disk graph with $R_i = r_i = 1$. They showed a polynomial-time 33-approximation for the problem. In this short note, we improve the result by reducing the performance ratio from 33 to 31.

2. Main Results

Consider the following greedy algorithm for vertex-coloring of double disk graph G:

Step 1. Put all vertices in a list as follows: At each iteration, choose a vertex with lowest degree and put it at the head of the list; then delete it from the graph. That is, all vertices of G are put in a list $v_1, \ldots, v_n$ such that for every i, $1 \leqslant i \leqslant n$, v_i has the least degree in subgraph induced by $\{v_1, \ldots, v_i\}$.

Step 2. Color all vertices as follows: At each iteration, color the head in the list with a smallest color (note: each color is represented by an integer) not appearing in its neighbors. Then delete it from the list. Therefore, vertex v_i is in color not bigger than one plus the degree of v_i in subgraph induced by $\{v_1, ..., v_i\}$.

Malesinska et al. [3] showed that this greedy algorithm has performance ratio 33. We improve it to 31 as follows.

THEOREM 1. *The above greedy algorithm is a polynomial-time 31-approximation for vertex-coloring of double disk graphs.*

To show this result, we first prove a lemma.

LEMMA 2. *For any double disk graph G, there exists a vertex with degree* $\leqslant 31\omega(G)-1$ *where* $\omega(G)$ *is the size of maximum clique in G.*

Proof. Choose a vertex v_i with smallest R_i. Without loss of generality, we may assume that $R_i=2$. Taking v_i as the center, draw 19 regular hexagons with edge length one as shown in Figure 1. For every two vertices v_j and v_k lying in the same hexagon, edge (v_j, v_k) must exist since $d(v_j, v_k) \leqslant 2 \leqslant R_j$. This means that all vertices lying in the same hexagon form a clique. To make the clique possibly larger for each outer hexagon, we use arc of outscribing circle to replace some edges. Meanwhile, we divide remaining area into 12 equal areas (see Figure. 1). Next, we show that for each of those 12 areas, all vertices adjacent to v_i, lying into the area, also form a clique.

To do so, we consider two points v_j and v_k lying in the same one of the 12 areas and assume $d(v_i, v_j) \leqslant d(v_k, v_i)$. We claim that $d(v_j, v_k) \leqslant \max(2, d(v_k, v_i)-2)$. To prove our claim, we first show three facts as follows:

FACT 1. Suppose v_h is on the extension of segment (v_j, v_k). If the claim is true for v_j and v_h, then it is true for v_j and v_k.

In fact, $d(v_h, v_k) \geqslant d(v_h, v_i) - d(v_k, v_i)$. Hence,

$$\begin{aligned} d(v_j, v_k) &= d(v_j, v_h) - d(v_k, v_h) \\ &\leqslant \max(2, d(v_i, v_h)-2) - d(v_i, v_h) + d(v_i, v_k) \\ &\leqslant \max(2, d(v_i, v_k)-2) \end{aligned}$$

FACT 2. Suppose $d(v_{j'}, v_k) \geqslant d(v_j, v_k)$ and the claim is true for $v_{j'}$ and v_k. Then it is true for v_j and v_k.

This fact is trivial. However, it has two important special cases:

(a) v_j' is on the extension of segment (v_k, v_j).
(b) v_j is on the segment $(v_{j'}, v_{j''})$ and the claim is true for $v_{j'}$ and v_k, and also true for $v_{j''}$ and v_k. In this case, we have either $d(v_j, v_k) \leqslant d(v_{j'}, v_k)$ or $d(v_j, v_k) \leqslant d(v_{j''}, v_k)$.

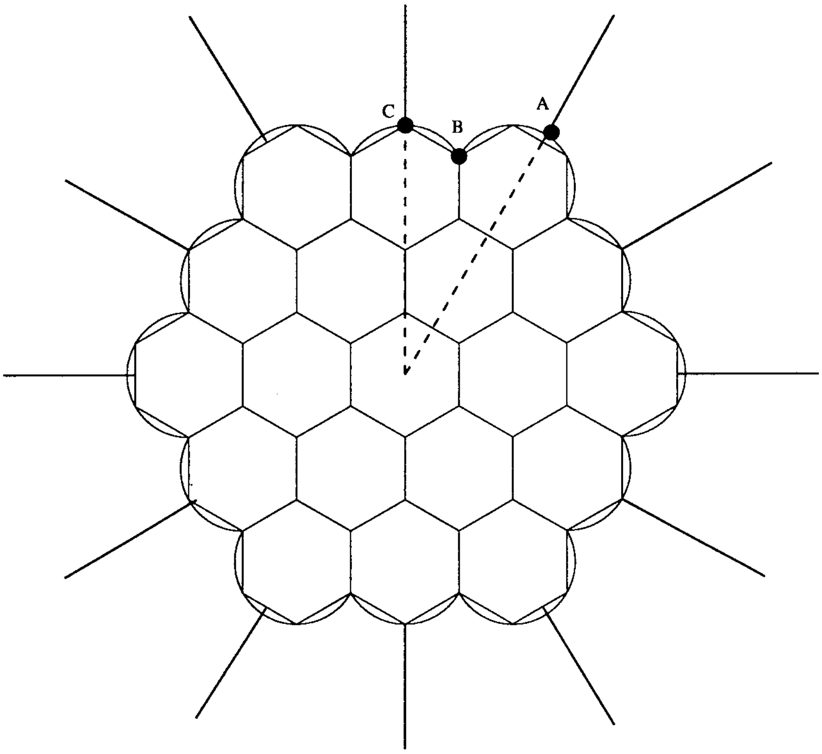

Figure 1. Thirty-one areas.

FACT 3. Suppose v_h is on the segment (v_i, v_k) with $d(v_i, v_h) \geqslant 4$ and the claim is true for v_j and v_h. Then the claim is true for v_j and v_k.

In fact, $d(v_h, v_k) \geqslant d(v_j, v_k) - d(v_j, v_h)$. Hence,

$$\begin{aligned} d(v_j, v_k) &\leqslant d(v_j, v_h) + d(v_h, v_k) \\ &\leqslant \max(2, d(v_i, v_h) - 2) + d(v_h, v_k) \\ &= d(v_i, v_h) - 2 + d(v_h, v_k) \\ &= \max(2, d(v_i, v_k) - 2) \end{aligned}$$

By Facts 1, 2 and 3, to show our claim true for v_j and v_k with $d(v_i, v_j) \leqslant d(v_i, v_k)$, it suffices to study the case that both v_j and v_k lie on the boundary of the area and one of the following holds:

(1) $d(v_i, v_j) = d(v_i, v_k)$.
(2) v_j lies at one of three convex corners A, B, and C as shown in Figure 1.

First, consider case (1). The following fact is important in this case.

FACT 4. Consider an angle $\angle v_j v_i v_k \leqslant 30°$. Let $v_{j'}$ be a point on segment (v_i, v_j) and $v_{k'}$ a point on segment (v_i, v_k). Suppose $d(v_i, v_{j'}) = d(v_i, v_{k'}) \geqslant 4$ and

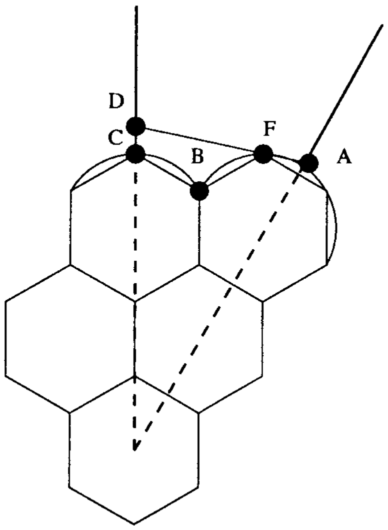

Figure 2. (A, D) intersects the boundary at F.

$d(v_i, v_j) = d(v_i, v_k)$. If the claim is true for $v_{j'}$ and $v_{k'}$, then it is also true for v_j and v_k.

In fact, $d(v_j, v_k) = d(v_{j'}, v_{k'}) + d(v_{k'}, v_k) \cdot 2\sin\angle v_j v_i v_k/2$. Therefore

$$\begin{aligned} d(v_j, v_k) &\leqslant d(v_{j'}, v_{k'}) + d(v_{k'}, v_k) \cdot 2\sin 15^\circ \\ &\leqslant \max(2, d(v_i, v_{k'}) - 2) + d(v_{k'}, v_k) \\ &= d(v_i, v_{k'}) + d(v_{k'}, v_k) - 2 \\ &= \max(2, d(v_i, v_k) - 2) \end{aligned}$$

Let D be a point on the extension of (v_i, C) such that $d(v_i, D) = d(v_i, A) = 1 + 2\sqrt{3}$. Let F be the intersection point of (A, D) and the arc boundary of the area (Figure 2). It is easily verified that F indeed is a vertex of a hexagon since $\angle v_i AD = 75^\circ$. For $d(v_i, v_j) = d(v_i, v_k) \geqslant d(v_i, F)$, by Facts 1 and 4 the truth of the claim for v_j and v_k can be derived from the truth of the claim for A and D. The latter can be verified by

$$d(A, D) = (1 + 2\sqrt{3}) \cdot 2\sin 15^\circ = (5 - \sqrt{3})/\sqrt{2} < 2\sqrt{3} - 1 = d(v_i, A) - 2.$$

For $d(v_i, v_j) = d(v_i, v_k) < d(v_i, F)$, it is easy to see that $d(v_j, v_k) < d(A, F) = 3\sqrt{2} - \sqrt{6} < 2$. Therefore, the claim is true in case (1).

Next, we consider case (2). For $v_j = A$, by Fact 3 it suffices to show the truth of the claim when $v_k = D$, which is already proved in the above. For $v_j = B$, by Fact 3 it suffices to show the truth of the claim when $v_k = A$ or C; in both subcases, $d(v_j, v_k) \leqslant 2$ and hence the claim is true. For $v_j = C$, by Fact 3 it suffices to show the truth of the claim when $v_k = A$; in this subcase, $d(v_j, v_k) = d(C, A) = \sqrt{5} < 2\sqrt{3} - 1$. This completes the proof of the claim.

Note that since v_k is adjacent to v_i, we must have $d(v_i, v_k) - 2 \leqslant R_k$. Since $R_i = 2$ is the smallest one, we also have $2 \leqslant R_k$. Therefore, $\max(2, d(v_i, v_k) - 2) \leqslant R_k$. It follows from the claim that $d(v_j, v_k) \leqslant R_k$. Therefore, all vertices adjacent to v_i and lying in the same area form a clique. Since thre are totally 31 areas, the degree of v_i is at most $31\omega(G)$.

To get a little smaller bound, we turn around our area-division around v_i and let one vertex lie on a boundary of areas. This trick can reduce the upper bound from $31\omega(G)$ to $31\omega(G) - 1$. □

Now, we note that the subgraph of double disk graph induced by a subset of vertices is still a double disk graph. Therefore, at each iteration of Step 2, the degree of vertex v_i is at most $31\omega(G) - 1$ in subgraph induced by $\{v_1, ..., v_i\}$. It follows that $31\omega(G)$ colors are enough to use in Step 2. Since $\omega(G)$ is a lower bound for optimal solution of vertex coloring, the greedy algorithm has performance ratio 31. This completes the proof of Theorem 1.

It is worth mentioning that Theorem 1 still holds if in Step 1, we order $v_1, v_2, ..., v_n$ to satisfy $R_1 \geqslant R_2 \geqslant \cdots \geqslant R_n$. This can be easily seen from the proof of Lemma 1.

3. Discussion

There are three important special cases for double disk graphs: unit disk graph ($r_i = R_i = 1$ for all i), intersection disk graph ($r_i = R_i$ for all i), and containment disk graph ($r_i = 0$ for all i). The vertex-coloring for those disk graphs has been studied extensively in the literature [2,4] due to its applications in wireless networks, including radio broadcast networks and cellular telephone networks. For unit disk graph G, the best known result is $\chi(G) \leqslant 3\omega(G) - 2$ [5] where $\chi(G)$ is the chromatic number of G, i.e., the minimum number of colors used for vertex-coloring. For intersection disk graph or containment disk graph G, it has been known that $\chi(G) \leqslant 6(\omega(G) - 1)$ [3].

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